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Poisson maps and integrable deformations of the Kowalevski top

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Abstract

We construct a Poisson map between manifolds with linear Poisson brackets corresponding to the Lie algebras $e(3)$ and $so(4)$. Using this map we establish a connection between the deformed Kowalevski top on $e(3)$ proposed by Sokolov and the Kowalevski top on $so(4)$. The connection between these systems leads to the separation of variables for the deformed system on $e(3)$ and yields the natural 5×5 Lax pair for the Kowalevski top on $so(4)$.

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1. Introduction

In 1888 Sophie Kowalevski [1] found and integrated a new integrable case of rotation of a heavy rigid body around a fixed point. In modern terms, this is an integrable system on the orbits of the Euclidean Lie algebra $e(3)$ with a quadratic and a quartic in angular momenta integrals of motion.

The Kowalevski top can be generalized in several directions. We can change either the initial phase space or the form of the Hamilton function. In 1981 Komarov [2] considered the Kowalevski top on $so(4)$, $e(3)$ and $so(3, 1)$ Lie algebras. The separation of variables for these generalizations was constructed in [3]. Recently in 2001, Sokolov has found integrable deformations of the Kowalevski Hamiltonian on $e(3)$ and $so(4)$ algebra [4, 6, 7]. A Lax representation for the deformed Kowalevski Hamiltonian on $e(3)$ was found in [5].

In this paper we establish an explicit nonlinear map of the Kowalevski top on $so(4)$ to the deformed Kowalevski case on $e(3)$. The connection between the systems leads to the separation of variables for the system on $e(3)$[3] and yields a natural 5×5 Lax pair for the Kowalevski top on $so(4)$ which was unknown. This Lax matrix provides an algebraic curve for

[3] Actually, we first found a separation of variables for this model and after that, comparing it with known separation of variables for the $so(4)$ Kowalevski top, found the Poisson map between $e(3)$ and $so(4)$.

the Kowalevski top on $so(4)$ different from the generalized Kowalevski curve [3] associated with the known separation of variables. For the deformed Kowalevski Hamiltonian on $so(4)$ neither a separation of variables nor Lax representation are found yet.

The existence of the Poisson map between $e(3)$ and $so(4)$ also allows us to construct a new $so(4)$ generalization of the Goryachev–Chaplygin top.

2. Deformations of the Kowalevski top

The rigid body motion about a fixed point under the influence of gravity is described by six dynamical variables—three components of the angular momentum $\mathbf{J} = (J_1, J_2, J_3)$ and three components of the gravity vector $\mathbf{x} = (x_1, x_2, x_3)$—everything with respect to a moving orthonormal frame attached to the body. The invariance under rotation about the direction of gravity leads to conservation of the angular momentum component along the gravity vector. When its value is fixed, the system is usually considered to have only two degrees of freedom [8] such that the Poisson sphere S^2 acts as a reduced configuration space. The reduced phase space may be identified with the coadjoint orbit of Euclidean $e(3)$ algebra with the Lie–Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k \qquad \{x_i, x_j\} = 0 \tag{2.1}$$

where ε_{ijk} is the totally skew-symmetric tensor. These brackets have two Casimir functions

$$A = \mathbf{x}^2 \equiv \sum_{k=1}^{3} x_k^2 \qquad B = (\mathbf{x} \cdot \mathbf{J}) \equiv \sum_{k=1}^{3} x_k J_k. \tag{2.2}$$

Fixing their values we obtain a generic symplectic leaf of $e(3)$

$$\mathcal{E}_{ab}: \quad \{\mathbf{x}, \mathbf{J} : A = aB = b\}$$

which is a four-dimensional symplectic manifold.

The Hamilton function for the original Kowalevski top is given by

$$H = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 x_1 \qquad c_1 \in \mathbb{C}. \tag{2.3}$$

This Hamiltonian and additional integral of motion

$$K = \xi \cdot \xi^* \tag{2.4}$$

where

$$\xi = (J_1 + \mathrm{i}J_2)^2 - 2c_1(x_1 + \mathrm{i}x_2) \qquad \xi^* = (J_1 - \mathrm{i}J_2)^2 - 2c_1(x_1 - \mathrm{i}x_2)$$

are in the involution and define the moment map whose fibres are Liouville tori in $\mathcal{E}_{ab}$.

The most general known deformation of the Hamiltonian (2.3) admitting quadratic and linear terms is defined by the following Hamiltonian:

$$\hat{H}_\varkappa = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 y_1 + 2c_2 J_3 y_2 - c_2^2 y_3^2 + 2c_3(J_3 + c_2 y_2) \qquad c_1, c_2, c_3 \in \mathbb{C} \tag{2.5}$$

(see equation (3.2) in [6]). The corresponding phase space is a generic orbit of the $so(4)$ Lie algebra with the Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k \qquad \{J_i, y_j\} = \varepsilon_{ijk} y_k \qquad \{y_i, y_j\} = \varkappa^2 \varepsilon_{ijk} J_k. \tag{2.6}$$

Notice that the deformation parameters are not only c_2 and c_3 in equation (2.5) but also $\varkappa$ entering the Lie algebra (2.6). Because physical quantities $\mathbf{y}, \mathbf{J}$ should be real, $\varkappa^2$ must be real too and algebra (2.6) is reduced to its two real forms $so(4, \mathbb{R})$ or $so(3, 1, \mathbb{R})$ for positive and negative $\varkappa^2$, respectively. For brevity we call it $so(4)$.

Fixing values a' and b' of the Casimir functions

$$A_\varkappa = \mathbf{y}^2 + \varkappa^2 \mathbf{J}^2 \qquad B_\varkappa = (\mathbf{y} \cdot \mathbf{J}) \tag{2.7}$$

we obtain a four-dimensional orbit of $so(4)$

$$\mathcal{O}_{a'b'}: \quad \{\mathbf{y}, \mathbf{J} : A_\varkappa = a', B_\varkappa = b'\}$$

which is the reduced phase space for the deformed Kowalevski top.

Performing a linear canonical transformation

$$\begin{array}{lll} J_1 \to J_1 & J_2 \to c_4(J_2 + c_2 y_3) & J_3 \to c_4(J_3 - c_2 y_2) \\ y_1 \to y_1 & y_2 \to c_4(y_2 + \varkappa^2 c_2 J_3) & y_3 \to c_4(y_3 - \varkappa^2 c_2 J_2) \end{array}$$

where $c_4 = \left(1 + \varkappa^2 c_2^2\right)^{-\frac{1}{2}}$, we reduce the Hamiltonian $\tilde{H}_\varkappa$ (2.5) to the following Hamilton function

$$\hat{H}_\varkappa = J_1^2 + \left(1 - \varkappa^2 c_2^2\right) J_2^2 + 2J_3^2 + 2c_1 y_1 + 2c_2(y_2 J_3 - y_3 J_2) + 2c_3 c_4^{-1} J_3 \tag{2.8}$$

which is linear in $\mathbf{y}$.

Parameter c_3 in equations (2.5) and (2.8) corresponds to the Kowalevski gyrostat [9, 6]. In this paper we consider the case $c_3 = 0$.

The integration procedure for Hamiltonian (2.3) proposed by Kowalevski is based on the fact that the additional integral of motion (2.4) is a product of two quadratic factors.

For the deformed Kowalevski top (2.8) the second integral of motion $\hat{K}_\varkappa$ can be written as

$$\hat{K}_\varkappa = \hat{\xi} \cdot \hat{\xi}^* + 4\varkappa^2 \left(\mathbf{J}^2 - J_2^2\right)\left(c_1^2 + c_2^2\left(\mathbf{J}^2 - J_2^2\right)\right) \tag{2.9}$$

where

$$\begin{aligned} \hat{\xi} &= \xi - c_2\{\mathbf{J}^2, y_1 + \mathrm{i} y_2\} - c_2^2\left(A_\varkappa - \varkappa^2 J_2^2\right) \\ \hat{\xi}^* &= \xi^* - c_2\{\mathbf{J}^2, y_1 - \mathrm{i} y_2\} - c_2^2\left(A_\varkappa - \varkappa^2 J_2^2\right). \end{aligned} \tag{2.10}$$

For the same integral we also have another useful representation

$$\hat{K}_\varkappa = \xi_\varkappa \cdot \xi_\varkappa^* + \varkappa^2 c_1^2 \left(2\hat{H}_\varkappa - \varkappa^2 c_1^2\right) + c_2 f(\mathbf{x}, \mathbf{J})$$

where

$$\xi_\varkappa = \xi + \varkappa^2 c_1^2 \qquad \xi_\varkappa^* = \xi^* + \varkappa^2 c_1^2.$$

and the polynomial $f(\mathbf{x}, \mathbf{J})$ can be easily restored from (2.9).

It follows from these formulae that the additional fourth degree integral of motion can be reduced to the product of two conjugated polynomials in the following two special cases

1. $c_2 = 0 \qquad K_\varkappa = \xi_\varkappa \cdot \xi_\varkappa^*$
2. $\varkappa = 0 \qquad \hat{K} = \hat{\xi} \cdot \hat{\xi}^*$

as well as for the original Kowalevski top. If $c_2 = 0$, the Hamiltonian for the Kowalevski top on the orbits of the Lie algebra $so(4)$ is given by the same formula (2.3):

$$H_\varkappa = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 y_1 \qquad c_1 \in \mathbb{C}. \tag{2.11}$$

The additional integral of motion $K_\varkappa = \xi_\varkappa \cdot \xi_\varkappa^*$ was found in [2]. A Lax pair of the Heine–Horozov [10] type and a separation of variables was constructed in [3].

For $\varkappa = 0$ the deformed Hamiltonian (2.8)

$$\hat{H} = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 x_1 + 2c_2(x_2 J_3 - x_3 J_2) \tag{2.12}$$

on $e(3)$ has been considered in [4, 6].

A Lax pair with a spectral parameter for the Kowalevski top has been found in [11]. Using this Lax representation and the standard finite-band integration technique, Bobenko *et al* [12] found explicit expressions for the solutions of the Kowalevski top which are much simpler than the original formulae of Kowalevski and Kötter. A Lax pair generalizing the corresponding result by Reyman and Semenov-Tian-Shansky was found by Sokolov and Tsiganov [5].

Below we present nonlinear Poisson maps between $e(3)$ and $so(4)$ Poisson manifolds. This allows us to relate various integrable systems on the different symplectic manifolds. As an example, we found an explicit mapping of integrable system (2.12) on $e(3)$ to the Kowalevski top (2.11) on $so(4)$. Using this, we find a Lax pair of Heine–Horozov type [10] and construct a separation of variables for the system with Hamiltonian (2.12) on $e(3)$ following [3]. On the other hand, using the results of [5] we construct a Lax pair for the Kowalevski top on $so(4)$.

3. Poisson maps of $e(3)$ and $so(4)$ manifolds

Let $\mathcal{M}_1$ be a Poisson manifold with generators $x_1, \ldots, x_n$ and Poisson bracket $\{\,,\}_1$, and $\mathcal{M}_2$ another Poisson manifold with generators $X_1, \ldots, X_m$ and Poisson bracket $\{\,,\}_2$. A map σ defined by

$$X_i = \Psi_i(\mathbf{x}) \qquad i = 1, \ldots, m \tag{3.1}$$

where $\mathbf{x} = (x_1, \ldots x_n)$, is called *Poisson map* (or Poisson homomorphism) if $\{\sigma(F), \sigma(G)\}_1 = \sigma(\{F, G\}_2)$ for any functions F and G on $\mathcal{M}_2$.

Example. Let p_i and x_j, $i, j = 1, 2, 3$ be canonical variables on manifold $\mathcal{M}_1$ with a Poisson bracket $\{p_i, x_j\}_1 = \delta_{ij}$, and let J_i, x_k form the manifold $\mathcal{M}_2$ with respect to a Poisson bracket of $e(3)$ Lie algebra: $\{J_i, J_j\}_2 = \varepsilon_{ijk} J_k$, $\{J_i, x_j\}_2 = \varepsilon_{ijk} x_k$, $\{x_i, x_j\}_2 = 0$ with Casimir elements $x_i x_i = a$ and $x_i J_i = b = 0$. Then the map $\sigma : \{\,,\}_1 \to \{\,,\}_2$ defined by $J_i = \epsilon_{ijk} x_j p_k$ establishes a Poisson map $\mathcal{M}_1 \to \mathcal{M}_2$.

If both Poisson brackets $\{\,,\}_1$ and $\{\,,\}_2$ are linear (and therefore related to some Lie algebras), the linear Poisson maps (3.1) correspond to homomorphisms of these Lie algebras.

If $\mathcal{M}_1$ coincides with $\mathcal{M}_2$, the Poisson maps are called *canonical transformations*. The problem of complete efficient description of all nonlinear canonical transformations is unsolvable. The reason is that for any function $f(\mathbf{x})$ the flow defined by ordinary differential equations (ODEs) $\mathbf{x}_t = \{f, \mathbf{x}\}_1$ yields a one-parameter group of canonical transformations. However, we can investigate some interesting subgroups of nonlinear canonical transformations.

In this paper we deal with linear Poisson brackets corresponding to the Lie algebras $e(3)$ and $so(4)$. For brevity we use the same notations for both the Poisson manifolds and the Lie algebras. In the next section we consider some special subgroups of nonlinear canonical transformations of $e(3)$.

3.1. Canonical transformations of $e(3)$

Consider the Poisson manifold $e(3)$ defined by linear brackets (2.1). Linear canonical transformations of $e(3)$ consist of rotations

$$\mathbf{x} \to \alpha U \mathbf{x} \qquad \mathbf{J} \to U \mathbf{J} \tag{3.2}$$

where α is an arbitrary parameter and U is an orthogonal constant matrix, and shifts

$$\mathbf{x} \to \mathbf{x} \qquad \mathbf{J} \to \mathbf{J} + S\mathbf{x} \tag{3.3}$$

where S is an arbitrary 3×3 skew-symmetric constant matrix.

Example 1. The composition of the scaling $\mathbf{x} \to \alpha\mathbf{x}$ and the rotation around third axis defined by

$$U = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

relates different orbits $\mathcal{E}_{a,b}$ and $\mathcal{E}_{\alpha^2 a,\alpha b}$ of $e(3)$ and changes the form of original Hamiltonian by

$$H \to \tilde{H} = J_1^2 + J_2^2 + 2J_3^2 + 2\tilde{c}_1 x_1 + 2\tilde{c}_2 x_2 \qquad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}. \tag{3.4}$$

Here $\tilde{c}_1 = \alpha c_1 \cos(\psi)$ and $\tilde{c}_2 = \alpha c_1 \sin(\psi)$.

Example 2. Transformation (3.3) with

$$S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

changes the form of the Hamiltonian (2.3) as follows:

$$H \to \tilde{H} = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 x_1 + 2(x_2 J_1 - x_1 J_2) + \left(x_1^2 + x_2^2\right).$$

This form of the Hamiltonian involves the third component of the vector product $\mathbf{J} \times \mathbf{x}$ and, at first glance, looks similar to the deformed Hamilton function (2.12), which contains the first component. However, transformations (3.2) and (3.3) are not enough to relate the deformed Kowalevski top (2.12) and the original Kowalevski top (2.3) on $e(3)$.

Let parameter α and matrices U and S in equations (3.2) and (3.3) be functions of the Casimir elements A, B. In this case, the transformations remain as Poisson mappings. Such Poisson maps change the form of the Hamiltonian as a function on the whole Poisson manifold. For instance, the Hamilton function (3.4) becomes

$$\tilde{H}(A, B) = J_1^2 + J_2^2 + 2J_3^2 + 2\tilde{c}_1(A, B)x_1 + 2\tilde{c}_2(A, B)x_2$$

where $\tilde{c}_1(A, B)$ and $\tilde{c}_2(A, B)$ are arbitrary functions on the Casimir elements (2.2). Of course, on each symplectic leaf the function $\tilde{H}(A, B)$ coincides with (3.4) and, therefore, the above construction of nonlinear Poisson mappings is trivial.

3.1.1. Generalized shifts of **J**. Consider the following generalizations of transformations (3.3):

$$\mathbf{x} \to \mathbf{x} \qquad \mathbf{J} \to \mathbf{J} + \mathbf{g}(\mathbf{x}) \tag{3.5}$$

where components $g_k(x_1, x_2, x_3)$ of the vector $\mathbf{g}$ are nonlinear functions of the Poisson vector $\mathbf{x}$. Substituting new variables into equation (2.1) we arrive at the following conditions on the vector $\mathbf{g}$

$$\operatorname{div} \mathbf{g} = 2\beta'(A) \qquad (\mathbf{x} \cdot \mathbf{g}) = \beta(A) \tag{3.6}$$

where β is an arbitrary function of the Casimir element $A = \mathbf{x}^2$.

Proposition 1. *A general solution of equations (3.6) is given by*

$$\mathbf{g} = \mathbf{x} \times (\operatorname{grad} W + F\mathbf{n}) + \beta\mathbf{f}$$

where potential $W(\mathbf{x})$ is an arbitrary scalar function of $\mathbf{x}$*, F is an arbitrary scalar function of two variables $x_1 + x_2 + x_3$, and $x_1^2 + x_2^2 + x_3^2$, and vectors* $\mathbf{n}$ *and* $\mathbf{f}$ *are given by*

$$\mathbf{n} = (1, 1, 1) \qquad \mathbf{f} = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2}, 0\right).$$

Under the transformation (3.5) the values of the Casimir functions are changed as

$$\tilde{a} = a \qquad \tilde{b} = b + \beta(a).$$

Thus, equation (3.5) is a nonlinear canonical transformation which relates the symplectic manifolds $\mathcal{E}_{ab}$ and $\mathcal{E}_{\tilde{a}\tilde{b}}$. We can apply this transformation in order to obtain 'new' integrable systems on these manifolds.

3.1.2. Generalized rotations. Consider generalized rotations of the form

$$\mathbf{x} \to \tilde{\mathbf{x}} = \alpha(\mathbf{x}, \mathbf{J})\, U(\mathbf{x}, \mathbf{J})\mathbf{x} \qquad \mathbf{J} \to \tilde{\mathbf{J}} = U(\mathbf{x}, \mathbf{J})\mathbf{J}$$

where the scalar factor $\alpha(\mathbf{x}, \mathbf{J})$ and the orthogonal matrix $U(\mathbf{x}, \mathbf{J})$ are some functions of variables $\mathbf{x}$ and $\mathbf{J}$. Requiring this transformation to be a Poisson map, we obtain a system of partial differential equations for $\alpha(\mathbf{x}, \mathbf{J})$ and $U_{ij}(\mathbf{x}, \mathbf{J})$. It would be interesting to find a general solution of this system. Here we consider a particular case when $\alpha(\mathbf{J})$ and $U_{ij}(\mathbf{J})$ depend on one (say, the third) component of angular momenta only.

Proposition 2. *Let $f(J_3)$ and $g(J_3)$ be any functions such that*

$$f^2 + g^2 = c^2 = \text{const},$$

then the mapping

$$\varphi : \mathbf{x} \to \sqrt{f^2 + g^2}\, U\mathbf{x} \qquad \mathbf{J} \to U\mathbf{J} \tag{3.7}$$

where

$$U = \frac{1}{\sqrt{f^2 + g^2}} \begin{pmatrix} f & g & 0 \\ -g & f & 0 \\ 0 & 0 & \sqrt{f^2 + g^2} \end{pmatrix}$$

is a canonical transformation of $e(3)$, which changes the values of Casimir functions (2.2) by the rule

$$\tilde{a} = ac^2 \qquad \tilde{b} = bc.$$

The generalized rotation (3.7) changes the form of the original Hamiltonian for the Kowalevski top (2.3) as follows:

$$H \to \tilde{H} = {J_1}^2 + {J_2}^2 + 2{J_3}^2 + x_1 f(J_3) + x_2 g(J_3).$$

In particular, with the help of such transformation we can obtain the following exotic Hamiltonian

$$\tilde{H} = {J_1}^2 + {J_2}^2 + 2{J_3}^2 + x_1 \sin(J_3) + x_1 \cos(J_3).$$

3.2. Poisson map between $so(4)$ and $e(3)$

In this section we consider Poisson maps between Poisson manifolds of $so(4)$ with generators $\tilde{J}_i, y_j$ and $e(3)$ with generators J_i, x_j. We restrict ourselves to special maps of the form

$$\tilde{\mathbf{J}} = \mathbf{J} \qquad \mathbf{y} = \alpha(A, B)\mathbf{x} + U(x_1, x_2, x_3, A, B)\mathbf{J}$$

where α is a scalar function of Casimir elements (2.2) and U is a matrix, which is not assumed to be orthogonal. Such maps identify the rotation subalgebras of $so(4)$ and $e(3)$.

The relations $\{J_i, y_j\} = \varepsilon_{ijk} y_k$ between components of the vectors $\mathbf{y}$ and $\mathbf{J}$ bring to an overdetermined system of partial differential equations for the matrix U. This system has the following general solution

$$U = \beta(A, B)\mathrm{I}_3 + \gamma(A, B) \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} + \delta(A, B) \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{pmatrix}$$

depending on arbitrary functions β, γ and δ of the Casimir elements (2.2), where I_3 is a unit 3×3 matrix. Notice that, if $\beta + A\delta = 1, \gamma^2 = (\beta + 1)\delta$, the above formula for U coincides with the well-known Gibbs representation [13] of an arbitrary orthogonal matrix.

The relations $\{y_i, y_j\} = \varepsilon_{ijk}\varkappa^2 J_k$ give rise to a system of algebraic equations for α, β, γ and $\varkappa$, which has only two different solutions. In the first case $\gamma = 0, \beta^2 = \varkappa^2, \alpha = -A\delta$ and the solution describes a reduction $e(3)$ to $so(3)$ by the trivial scaling $\mathbf{y} = \varkappa\mathbf{J}$.

The second solution is

$$\beta = 0 \qquad \gamma^2 = -\frac{\varkappa^2}{A}$$

and δ is an arbitrary function which can be removed by shift $\alpha \to \alpha + A\delta$. Thus, this solution corresponds to the transformation

$$\zeta : \quad \mathbf{J} \to \mathbf{J} \qquad \mathbf{y} = \alpha\mathbf{x} + \gamma\mathbf{x} \times \mathbf{J} \tag{3.8}$$

which maps the manifold $e(3)$ to the manifold $so(4)$. Below we consider a special case $\alpha = \text{const}$ in more detail.

Proposition 3. *Suppose $\alpha \neq 0$ is a constant and γ is a solution of equation*

$$A\gamma^2 + \varkappa^2 = 0. \tag{3.9}$$

Then transformation (3.8) is a Poisson map of $e(3)$ to $so(4)$.

The inverse Poisson map $so(4) \to e(3)$ is given by

$$\mathbf{x} = \frac{\alpha^2\mathbf{y} + \gamma_\varkappa^2(\mathbf{y} \cdot \mathbf{J})\mathbf{J} + \alpha\gamma_\varkappa(\mathbf{y} \times \mathbf{J})}{\alpha\left(\alpha^2 + \gamma_\varkappa^2\mathbf{J}^2\right)}, \tag{3.10}$$

where the algebraic function $\gamma_\varkappa(A_\varkappa, B_\varkappa)$ depending on $so(4)$-Casimir elements (2.7) is defined by

$$B_\varkappa^2\gamma_\varkappa^4 + A_\varkappa\alpha^2\gamma_\varkappa^2 + \alpha^4\varkappa^2 = 0. \tag{3.11}$$

Notice that the branches of the square roots in equations (3.9) and (3.11) have to be consistent.

The Poisson maps (3.8) and (3.10) give rise to the symplectic correspondence between the symplectic submanifolds $\mathcal{E}_{ab}$ in $e(3)$ and symplectic submanifolds $\mathcal{O}_{a'b'}$ in $so(4)$, where

$$a' = \alpha^2 a + \frac{\varkappa^2 b^2}{a} \qquad b' = \alpha b. \tag{3.12}$$

Obviously, compositions of the Poisson maps (3.8) and (3.10) with canonical transformations of $e(3)$ or $so(4)$ give rise to different Poisson maps relating $e(3)$ and $so(4)$.

The singular points of the transformation can be easily seen from equations (3.8)–(3.10).

It turns out that the Poisson maps (3.8) and (3.10) establish a correspondence between the reduced four-dimensional phase spaces of the Kowalevski top on $so(4)$ and the deformed Kowalevski top on $e(3)$:

Theorem 1. *Transformation (3.8) sends the Hamilton function*

$$H_\varkappa = J_1^2 + J_2^2 + 2J_3^2 + 2\tilde{c}_1 y_1$$

on $\mathcal{O}_{a'b'}$ to the Hamilton function

$$\hat{H} = J_1^2 + J_2^2 + 2J_3^2 + 2c_1x_1 + 2c_2(x_2J_3 - x_3J_2) \tag{3.13}$$

on $\mathcal{E}_{ab}$, where $c_1 = \alpha\tilde{c}_1$, $c_2 = \gamma\tilde{c}_1$ and the constants a, b, a', b' are related by equation (3.12).

Notice that c_1 in formula (3.13) is a constant whereas c_2 is a function of the Casimir element A. However, on each symplectic leaf c_1 and c_2 are constants and $\hat{H}$ from equation (3.13) coincides with equation (2.12).

Remark. In [14] a different Poisson map

$$\mathbf{J} \to \mathbf{J} \qquad \mathbf{y} \to \mathbf{x} = \frac{\mathbf{J} \times (\mathbf{y} \times \mathbf{J})}{|\mathbf{J} \times (\mathbf{y} \times \mathbf{J})|}$$

from $so(4)$ to $e(3)$ was considered. This mapping takes any symplectic leaf $\mathcal{O}_{a'b'}$ of $so(4)$ to the same symplectic leaf $(\mathbf{x}, \mathbf{J}) = 0, \mathbf{x}^2 = 1$ of $e(3)$ and therefore it is not invertible. This mapping allows us to lift integrable Hamiltonians from $e(3)$ to $so(4)$ but it involves radicals and does not preserve the property of the Hamiltonians to be rational.

4. Lax representation for the $so(4)$ Kowalevski top

A Lax representation

$$\frac{\mathrm{d}}{\mathrm{d}t}L = [M, L] \tag{4.1}$$

for the Kowalevski top (2.3) was found by Reyman and Semenov-Tian-Shansky [11]. The corresponding Lax matrices are

$$L(\lambda) = \begin{pmatrix} 0 & J_3 & -J_2 & \lambda & 0 \\ -J_3 & 0 & J_1 & 0 & \lambda \\ J_2 & -J_1 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & -J_3 \\ 0 & \lambda & 0 & J_3 & 0 \end{pmatrix} - \frac{c_1}{\lambda}\begin{pmatrix} 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 & 0 \\ x_1 & x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\stackrel{\mathrm{def}}{=} \lambda\mathcal{A} + \sum_{i=1}^{3} J_i \cdot \mathcal{J}_i - \frac{c_1}{\lambda}\sum_{i=1}^{3} x_i \cdot \mathcal{X}_i \tag{4.2}$$

and

$$M(\lambda) = 2\begin{pmatrix} 0 & -2J_3 & J_2 & -\lambda & 0 \\ 2J_3 & 0 & -J_1 & 0 & -\lambda \\ -J_2 & J_1 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

The characteristic curve $\mathrm{Det}(L(\lambda) - \mu \cdot I) = 0$, where $I = \mathrm{diag}(1, 1, 1, 1, 1)$ is the unit matrix, which provides a complete set of first integrals for the Kowalevski top [12].

It is essential for the general group-theoretical approach to integrable systems [11] that the matrices $\mathcal{A}, \mathcal{J}_i, \mathcal{X}_i$ belong to the matrix realization of the Lie algebra $so(3, 2)$ by 5×5 matrices $\mathcal{Z}$ satisfying the identity

$$\mathcal{Z}^T = -I_{3,2}\mathcal{Z}I_{3,2} \tag{4.4}$$

where $I_{3,2} = \mathrm{diag}(1, 1, 1, -1, -1)$. The Lax matrices (4.2) are invariant with respect to the following involution

$$\tau : \quad Z(\lambda) \to -Z^T(-\lambda).$$

Using the well-known isomorphism $so(3,2) \simeq sp(4,\mathbb{R})$ we can also obtain a 4×4 Lax pair for the Kowalevski top [12].

A Lax representation for the deformed Kowalevski top on $e(3)$ with the Hamilton function (2.12) was found in [5]. This representation involves an additional matrix

$$Y = \begin{pmatrix} 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 & 0 \\ -x_1 & -x_2 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \stackrel{\text{def}}{=} \sum_{i=1}^{3} x_i \mathcal{Y}_i .$$

The constant matrices $\mathcal{Y}_i$ are symmetrized anticommutators of matrix coefficients of the initial Lax matrix L (4.2)

$$\mathcal{Y}_i = \varepsilon_{ijk}(\mathcal{X}_j\mathcal{J}_k + \mathcal{J}_k\mathcal{X}_j).$$

They do not respect involution (4.4) and hence do not belong to the algebra $so(3,2)$.

Proposition 4 (Sokolov and Tsiganov [5])**.** *The flow with the Hamiltonian $\hat{H}$ (2.12) is equivalent to the matrix differential equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}_i(\lambda) = \hat{L}_i(\lambda)\hat{M}(\lambda) + \hat{M}^T(-\lambda)\hat{L}_i(\lambda) \qquad i = 1, 2 \tag{4.5}$$

where

$$\hat{L}_1(\lambda) = L(\lambda) + \frac{c_2}{2}\sum_{i=1}^{3} x_i \cdot ((\mathcal{X}_i - \mathcal{Y}_i)\mathcal{A} - \mathcal{A}(\mathcal{X}_i + \mathcal{Y}_i)) \qquad \hat{L}_2(\lambda) = -I + \frac{c_2}{\lambda}Y \tag{4.6}$$

$$\hat{M} = M + 2c_2 \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & -x_2 & 0 \end{pmatrix}$$

and the superscript T denotes matrix transposition.

It is easy to verify that the matrices $\hat{L}_{1,2}$ (4.6) can be rewritten as follows

$$\hat{L}_1 = (I - g^\tau)^{-1}L(I - g) + V \qquad \hat{L}_2 = -(I - g^\tau)^{-1}(I + g^\tau g)(I - g) \tag{4.7}$$

where

$$g = \frac{c_2}{\lambda}\begin{pmatrix} 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V = -\frac{c_2}{\lambda}\begin{pmatrix} 0 & 0 & 0 & (\mathbf{x}\times\mathbf{J})_1 & 0 \\ 0 & 0 & 0 & (\mathbf{x}\times\mathbf{J})_2 & 0 \\ 0 & 0 & 0 & (\mathbf{x}\times\mathbf{J})_3 & 0 \\ (\mathbf{x}\times\mathbf{J})_1 & (\mathbf{x}\times\mathbf{J})_2 & (\mathbf{x}\times\mathbf{J})_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the matrix V depends on the components $(\mathbf{x}\times\mathbf{J})_i$ of the cross product $\mathbf{x}\times\mathbf{J}$ only.

Relations (4.5) imply that matrices

$$\hat{L}_+ = \hat{L}_1(\lambda)\hat{L}_2^{-1}(\lambda) \qquad \hat{L}_- = \hat{L}_2^{-1}(\lambda)\hat{L}_1(\lambda) \tag{4.8}$$

satisfy the usual Lax equations (4.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}_+ = [\hat{L}_+, -\hat{M}^T(-\lambda)] \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{L}_- = [\hat{L}_-, \hat{M}(\lambda)].$$

The explicit form of the Lax matrices (4.8) is rather complicated. However, matrices $\hat{L}_\pm$ can be simplified with the help of a gauge transformation. Let us define a new matrix $\hat{L}$ by the formula

$$\hat{L}(\lambda) = -(I-g)\hat{L}_-(I-g)^{-1}.$$

Using equation (4.7) and the following property of V

$$(I-g^\tau)V(I-g)^{-1} = V$$

this matrix can be rewritten in the form

$$\hat{L}(\lambda) = (I+g^\tau g)^{-1}(L(\lambda)+V).$$

It can be verified that

$$\hat{L}^\tau(\lambda) = -(I-g^\tau)\hat{L}_+(I-g^\tau)^{-1}.$$

The next statement describes a Lax pair for the deformed Kowalevski top on $e(3)$ with Hamiltonian (2.12) related to the matrix $\hat{L}$.

Proposition 5. *The flow with the Hamiltonian $\hat{H}$ (2.12) is equivalent to the Lax equation (4.1), where*

$$\hat{L}(\lambda) = (I+g^\tau g)^{-1}(L(\lambda)+V) \qquad \text{and} \qquad \hat{M}(\lambda) = M(\lambda)(I+g^\tau g). \tag{4.9}$$

It is important that the product $g^\tau g$ depends on the Casimir function only:

$$g^\tau g = \frac{c_2^2\mathbf{x}^2}{\lambda^2}\mathcal{G} \qquad \mathcal{G} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{pmatrix}. \tag{4.10}$$

Taking this formula into account, we obtain

$$\hat{L}(\lambda) = \left(I - \frac{c_2^2\mathbf{x}^2}{\lambda^2+c_2^2\mathbf{x}^2}\mathcal{G}\right)\left(\lambda\mathcal{A} + \sum_{i=1}^3 J_i\cdot\mathcal{J}_i - \frac{1}{\lambda}\sum_{i=1}^3 y_i\cdot\mathcal{X}_i\right)$$

where $y_i = c_1x_i + c_2(\mathbf{x}\times\mathbf{J})_i$. We see that, for $c_2 = 0$, the matrix $\hat{L}$ coincides with equation (4.2).

Thus, in order to construct the Lax matrices (4.9) for the deformed Kowalevski top on $e(3)$ we have to substitute y_i instead of x_i into the Lax matrices found by Reyman and Semenov-Tian-Shansky [11] and multiply the result by matrices depending on the Casimir element only.

The fact that $\hat{L}$ depends only on variables J_i and $y_i = c_1x_i + c_2(\mathbf{x}\times\mathbf{J})_i$, which define transformation (3.8), allows us to construct a Lax representation for the Kowalevski top on $so(4)$. Namely, an obvious combination of propositions 3 and 5 leads to:

Theorem 2. *The matrices*

$$L_\varkappa(\lambda) = \left(I + \frac{\tilde c_1^2\varkappa^2}{\lambda^2 - \tilde c_1^2\varkappa^2}\mathcal{G}\right) \times \left[\begin{pmatrix} 0 & J_3 & -J_2 & \lambda & 0 \\ -J_3 & 0 & J_1 & 0 & \lambda \\ J_2 & -J_1 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & -J_3 \\ 0 & \lambda & 0 & J_3 & 0 \end{pmatrix} - \frac{\tilde c_1}{\lambda}\begin{pmatrix} 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & y_2 & 0 \\ 0 & 0 & 0 & y_3 & 0 \\ y_1 & y_2 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right] \tag{4.11}$$

and

$$M_\varkappa = 2\begin{pmatrix} 0 & -2J_3 & J_2 & -\lambda & 0 \\ 2J_3 & 0 & -J_1 & 0 & -\lambda \\ -J_2 & J_1 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \end{pmatrix}\left(I - \frac{\tilde c_1^2\varkappa^2}{\lambda^2}\mathcal{G}\right) \tag{4.12}$$

where $\mathcal{G}$ is given by (4.10), define a Lax representation for the $so(4)$ Kowalevski top with the Hamiltonian

$$H_\varkappa = J_1^2 + J_2^2 + 2J_3^2 + 2\tilde c_1 y_1 \qquad \tilde c_1 \in \mathbb{C}.$$

The characteristic curve $\mathrm{Det}(L_\varkappa(\lambda) - \mu I) = 0$ *provides a complete set of first integrals of motion*

$$\left(\tilde c_1^2\varkappa^2 - \lambda^2\right)\mu^4 + \mu^2\left(2\lambda^4 - \left(H_\varkappa + \tilde c_1^2\varkappa^2\right)\lambda^2 + \tilde c_1^2 A_\varkappa\right) = \lambda^6 - H_\varkappa\lambda^4 - K_\varkappa\lambda^2 - \tilde c_1^2 B_\varkappa^2.$$

It is seen that the Lax pair (4.11) and (4.12) on $so(4)$ is formed from the Lax pair (4.2) and (4.3) on $e(3)$ by substitution $x_i \to y_i \quad c \to \tilde c$ and multiplication by the λ-meromorphic diagonal constant matrix factors from the left and right, respectively:

$$\begin{aligned} L_\varkappa(\lambda) &= \left(I + \frac{\tilde c_1^2\varkappa^2}{\lambda^2 - \tilde c_1^2\varkappa^2}\mathcal{G}\right) L(\lambda)|_{x_i\to y_i, c_1\to\tilde c_1} \\ M_\varkappa(\lambda) &= M(\lambda)|_{c_1\to\tilde c_1}\left(I - \frac{\tilde c_1^2\varkappa^2}{\lambda^2}\mathcal{G}\right). \end{aligned} \tag{4.13}$$

This Lax pair for the Kowalevski top on $so(4)$ in principle allows us to apply the finite-band integration technique.

The following comments are in order.

- Multiplying $L_\varkappa$ by the factor $\lambda^2 - \tilde c_1^2\varkappa^2$, we can remove the poles at $\lambda = \pm\tilde c_1\varkappa$. Nevertheless just operator $L_\varkappa$ tends to the original Lax matrix from [11] as $\varkappa \to 0$. Probably this means that the poles in the Lax matrix for the Kowalevski top on $so(4)$ are essential.
- Substituting the Lax matrices for the Kowalevski gyrostat on $e(3)$ (see [11]) for L and M in equation (4.13), we obtain a Lax pair for the Kowalevski gyrostat on $so(4)$.
- The matrices $\hat L_{1,2}$ and $\hat L$ do not respect the involution (4.4) and, therefore, are out of the matrix realization of the Lie algebra $so(3,2)$. They cannot be rewritten as 4×4 matrices via the isomorphism $so(3,2) \simeq sp(4)$. Nevertheless precisely the matrices $\hat L_{1,2}$ and $\hat L$ provide a multi-dimensional generalization of the Kowalevski gyrostat [5].

Applying the Poisson map from proposition 3 to a similar Lax matrix for the Lagrange top on $e(3)$ [11], we obtain the following:

Proposition 6. *For the Lagrange top on $so(4)$ defined by the Hamiltonian*

$$H_\varkappa^{\mathrm{Lag}} = J_1^2 + J_2^2 + J_3^2 + 2cy_1 \qquad c \in \mathbb{C}$$

a Lax matrix is given by

$$L_\varkappa^{\mathrm{Lag}}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 + \dfrac{c^2\varkappa^2}{\lambda^2 - c^2\varkappa^2} \end{pmatrix} \begin{pmatrix} 0 & J_3 & -J_2 & \lambda - \dfrac{cy_1}{\lambda} \\ -J_3 & 0 & J_1 & -\dfrac{cy_2}{\lambda} \\ J_2 & -J_1 & 0 & -\dfrac{cy_3}{\lambda} \\ \lambda - \dfrac{cy_1}{\lambda} & -\dfrac{cy_2}{\lambda} & -\dfrac{cy_3}{\lambda} & 0 \end{pmatrix}.$$

The corresponding characteristic curve

$$(c^2\varkappa^2 - \lambda^2)\mu^4 - \left(\lambda^4 - H_\varkappa^{\mathrm{Lag}}\lambda^2 + c^2 A_\varkappa\right)\mu^2 = \left(K_\varkappa^{\mathrm{Lag}}\lambda^2 - cB_\varkappa\right)^2$$

where $K_\varkappa^{\mathrm{Lag}} = J_1$, provides a complete set of integrals of motion.

5. Separation of variables

The separation of variables for the Hamilton function $H_\varkappa$ (2.11) on $so(4)$ was obtained in [3] by a non-canonical reduction to the Neumann system. (Previously, in unpublished calculations Komarov obtained the result by a variant of the original Kowalevski approach.) The results of [3] were based on the fact that the evolutionary equations for the Kowalevski top, written in special variables of Haine and Horozov [10], coincide with the Neumann system.

Applying propositions 1 and 2 we can derive explicit formulae for the separation of variables for the model (2.12) on $e(3)$ from [3]. But historically we have obtained these formulae following the original Kowalevski work [1]. Below we follow this line.

Consider the Hamiltonian on $e(3)$

$$\hat{H} = J_1^2 + J_2^2 + 2J_3^2 + 2c_1x_1 + 2c_2(x_2J_3 - x_3J_2). \tag{2.13}$$

It is easy to prove that variables

$$z_1 = J_1 + \mathrm{i}J_2 \qquad z_2 = J_1 - \mathrm{i}J_2,$$

satisfy the following system of equations

$$\dot{z}_1^2 - F(z_1) + \hat{\xi}(z_1 - z_2)^2 = 0 \qquad \dot{z}_2^2 - F(z_2) + \hat{\xi}^*(z_1 - z_2)^2 = 0$$

where $\hat{\xi}, \hat{\xi}^*$ are given by equation (2.10) with $\varkappa = 0$:

$$\hat{\xi} = \xi - c_2\{\mathbf{J}^2, x_1 + \mathrm{i}x_2\} - c_2^2 A \qquad \hat{\xi}^* = \xi^* - c_2\{\mathbf{J}^2, x_1 - \mathrm{i}x_2\} - c_2^2 A.$$

Here $F(z)$ is a polynomial of four degree with coefficients being integrals of motion

$$F(z) = z^4 - 2\hat{H}z^2 + 8c_1Bz + \hat{K} - 4Ac_1^2 + 2c_2^2(2B^2 - \hat{H}A) - c_2^4A^2.$$

According to [1], we define the biquadratic form

$$F(z_1, z_2) = \tfrac{1}{2}\left(F(z_1) + F(z_2) - \left(z_1^2 - z_2^2\right)^2\right)$$

and the separated variables

$$s_{1,2} = \frac{F(z_1, z_2) \pm \sqrt{F(z_1)F(z_2)}}{2(z_1 - z_2)^2} \tag{5.1}$$

such that

$$\dot{s}_1 = \frac{\sqrt{P_5(s_1)}}{s_1 - s_2} \qquad \dot{s}_2 = \frac{\sqrt{P_5(s_2)}}{s_2 - s_1} \qquad P_5(s) = P_3(s)P_2(s). \tag{5.2}$$

Here $P_3(s)$ and $P_2(s)$ are polynomials of third and second degree:

$$P_3(s) = s\big(4s^2 + 4s\hat{H} + \hat{H}^2 - \hat{K} + 4c_1^2 A + 2c_2^2(\hat{H}A - 2B^2) + c_2^4 A^2\big) + 4c_1^2 B^2$$
$$P_2(s) = 4s^2 + 4(\hat{H} + c_2^2 A)s + \hat{H}^2 - \hat{K} + 2c_2^2 \hat{H} A + c_2^4 A^2.$$

To integrate equations (5.2) we should substitute the values of integrals of motion and Casimir elements found from initial data. Equations (5.2) are integrated in terms of genus two hyperelliptic functions of time.

As well as in the case of initial Kowalevski top [15] we can check by direct computations that functions $s_{1,2}$ (5.1) defined on the whole phase space commute with respect to initial Poisson brackets (2.1)

$$\{s_1, s_2\} = 0.$$

The reasons why the functions s_1, s_2 give rise to canonical variables on $e(3)$ seem to be unclear (see comments in [15], where the Poisson commutativity of s_1 and s_2 was originally pointed out).

The momenta $p_{1,2}$ conjugated to coordinates $s_{1,2}$ can be introduced according to [15] (see [3] for another approach). The result is

$$p_i = \frac{1}{4\sqrt{s_i}} \ln\left(\frac{2\sqrt{s_i P_5(s_i)} - P_3(s_i) - s_i P_2(s_i)}{4(as_i + b^2)\big(c_1^2 - c_2^2 s_i\big)}\right). \tag{5.3}$$

In variables (5.1) and (5.3) the Hamilton function (2.12) is given by

$$\hat{H} = -s_1 - s_2 + \frac{c_1^2 b^2}{2s_1 s_2} - \frac{ac_2^2}{2} + \frac{d_1 \cosh(4p_1\sqrt{s_1}) - d_2 \cosh(4p_2\sqrt{s_2})}{2(s_1 - s_2)}$$

where

$$d_i = \frac{\big(c_2^2 s_i - c_1^2\big)(as_i + b^2)}{s_i}.$$

Using this relation, we obtain two separated equations

$$2s_i^3 + \big(2\hat{H} + c_2^2 a\big)s_i^2 - \kappa s_i + c_1^2 b^2 = (c_2^2 s_i - c_1^2)(as_i + b^2)\cosh(4p_i\sqrt{s_i}), \tag{5.4}$$

where

$$4\kappa = \big(\hat{H} + c_2^2 a\big)^2 - \hat{K} + 2c_1^2 a.$$

As usual, canonical variables s_i (5.1) and p_i (5.3) are defined up to arbitrary canonical transformations that mix together s_i and p_i with the same i. Evidently, such transformations change the form of separated equations. Equations (5.4) coincide with the separated equations from [3] up to a canonical scaling of p_i and s_i. The mapping from [3] between the $so(4, \mathbb{C})$ Kowalevski top and the Neumann system relates the separated variables s_1, s_2 of Kowalevski top and separated variables $\lambda_{1,2}$ for the Neumann top by $s_{1,2} = 2\lambda_{1,2} + H$.

6. Summary

We present a Poisson map which relates rank-two Poisson manifolds $e(3)$ and $so(4)$. Using such transformations in rigid body dynamics, we obtain new results on deformations of the $e(3)$ and $so(4)$ Kowalevski tops.

The same Poisson map can be applied to other integrable systems, for instance to the deformation of the Goryachev–Chaplygin gyrostat proposed in [5]. In this case, mapping equations (3.8)–(3.10) sends the integrable Hamilton function on $e(3)$ [5]

$$H^g = J_1^2 + J_2^2 + 4J_3^2 + 2\alpha c_1 x_1 + 2\gamma c_1(x_2 J_3 - 2x_3 J_2)$$

to the following function on $so(4)$ manifold

$$H^g_\varkappa = J_1^2 + J_2^2 + 4J_3^2 - 2c_1 y_1 + \frac{2\varkappa^2 c_1 J_3}{\gamma \mathbf{y}^2}(\alpha y_2 + \gamma(y_1 J_3 - y_3 J_1))$$

where $A_\varkappa \gamma^2 + \alpha^2\varkappa^2 = 0$. This Hamiltonian commutes with

$$K^g_\varkappa = c_1 y_3 J_1 + \left(J_1^2 + J_2^2 - \frac{\varkappa^2 c_1 J_2}{\gamma \mathbf{y}^2}(\alpha y_3 - \gamma(y_1 J_2 - y_2 J_1))\right) J_3$$

on a special level of the Casimir function $B_\varkappa = 0$. Another version of the Goryachev–Chaplygin top on $so(4)$ was proposed in [14].

One of our main results is a Lax representation for the Kowalevski top on $so(4)$ provided by theorem 2. The Lax matrix $L_\varkappa(\lambda)$ (4.13) generates an algebraic curve different from the original Kowalevski one. Matrix $L_\varkappa(\lambda)$ should originate separated variables which in turn differ from those considered in section 5. Such separation of variables remains an open question as well as for the original $e(3)$ Kowalevski case.

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